

A JSJ splitting for triangulated open 3-manifolds

Sylvain Maillot

February 11, 2008

Abstract

We give a sufficient condition for an open 3-manifold to admit a decomposition along properly embedded open annuli and tori, generalizing the toric splitting of Jaco-Shalen and Johannson.

1 Introduction

This paper is a continuation of the program started in [8], whose goal is to find geometric conditions under which an open 3-manifold admits a canonical decomposition. In this paper we are concerned with Jaco-Shalen-Johannson (JSJ) splittings.

Let us first recall some standard terminology from 3-manifold theory. Throughout the paper we work in the PL category. A 3-manifold is *irreducible* if every embedded 2-sphere bounds a 3-ball. If M is an orientable 3-manifold and $F \subset M$ an embedded orientable surface not homeomorphic to S^2 , then F is *incompressible* if the homomorphism $\pi_1 F \rightarrow \pi_1 M$ induced by inclusion is injective. A 3-manifold M is *atoroidal* if all incompressible tori in M are boundary-parallel. It is *Seifert fibered*, or a *Seifert manifold*, if it fibers over a 2-dimensional orbifold.

Let M be an orientable, irreducible 3-manifold without boundary. When M is compact, Jaco-Shalen [4] and Johannson [5] found a canonical family of pairwise disjoint, embedded, incompressible 2-tori which split M into submanifolds that are either Seifert fibered or atoroidal. This family, called the *JSJ-splitting* of the 3-manifold M , has several additional properties; for instance, any incompressible torus embedded in M can be homotoped into a Seifert piece.

A simple way to construct the JSJ-splitting of a closed manifold was given in [9]: let $F_1, F_2 \subset M$ be two embedded surfaces in M . We say that F_1 is

homotopically disjoint from F_2 if there exists a surface F'_1 which is homotopic to F_1 and disjoint from F_2 . An incompressible torus is called *canonical* if it is homotopically disjoint from any other incompressible torus. Then when M is closed, its JSJ-splitting can be constructed by simply taking a disjoint collection of representatives of each homotopy class of canonical tori.

We want to extend this theory to noncompact manifolds. There are of course obvious sufficient conditions under which the theory goes through, for instance if M is homeomorphic to the interior of a compact manifold. Thus we are typically interested in open 3-manifolds with infinite topological complexity. Consider the following class of examples: let X be any orientable, irreducible, atoroidal 3-manifold whose boundary is an incompressible open annulus, different from $S^1 \times \mathbf{R} \times [0, \infty)$. Let Y be the product of S^1 with an orientable surface of infinite genus and boundary a line. Then by gluing X and Y together along their boundary annuli, one obtains a 3-manifold M . Such a manifold certainly has infinite topological complexity in any reasonable sense of the word (e.g. its fundamental group is infinitely generated). Yet it has an obvious splitting along a properly embedded incompressible annulus into an atoroidal piece X and a Seifert piece Y .

If in addition, X contains no properly embedded compact annuli other than those parallel to the boundary, then the Seifert submanifold Y will be ‘maximal’ in the sense that every incompressible torus can be homotoped into Y . Thus it seems to the author that this splitting qualifies as a JSJ-splitting. Note that such a manifold M contains no canonical tori; by contrast, it contains plenty of noncanonical tori, which in a sense ‘fill up’ the Seifert piece of the JSJ-splitting. This motivates the following definition:

Definition. A *JSJ-splitting* of an open, orientable, irreducible 3-manifold M is a locally finite collection \mathcal{C} of disjoint properly embedded incompressible tori and open annuli satisfying the following conditions:

- i. Each component of M split along \mathcal{C} is Seifert fibered or atoroidal.
- ii. Each torus of \mathcal{C} is canonical.
- iii. Every canonical torus is homotopic to a torus of \mathcal{C} .
- iv. Every incompressible torus is homotopic into a Seifert piece of M split along \mathcal{C} .

Although it may seem plausible at first sight that every open 3-manifold has a JSJ-splitting, this is not true. Indeed, in [6] there is a construction of a 3-manifold M_3 containing an infinite collection of pairwise non-homotopic canonical tori that intersect some fixed compact set essentially; therefore,

any collection of tori containing a representative of each homotopy class of canonical tori fails to be locally finite, however the representatives are chosen.

Our main result is a sufficient condition for an open 3-manifold M to admit a JSJ-splitting. Before stating it, we give a few definitions and conventions. We shall assume all 3-manifolds in this paper to be orientable.

Let M be a 3-manifold. If F_1 and F_2 are not homotopically disjoint, we will say that they *intersect essentially*. We say that F_1, F_2 *intersect minimally* if they are in general position, and the number of components of $F_1 \cap F_2$ cannot be reduced by homotoping one of the surfaces.

Note that by a theorem of Waldhausen, homotopy implies isotopy for compact surfaces in a Haken manifold M . This result is still true if M is open and irreducible because any homotopy takes place in a compact subset of M , which can be embedded into a Haken submanifold of M . Hence we shall sometimes use the words “homotopic” and “isotopic” interchangeably.

Let T be an incompressible torus in M . Assume that T is not canonical, and M is open. We shall see (Proposition 3.1 below) that there is a unique free isotopy class $\xi(T)$ of simple closed curves in T such that if T' is an incompressible torus which is not homotopically disjoint from T and intersects T' minimally, then $T \cap T'$ consists of curves belonging to $\xi(T)$. We say that a simple closed curve in T is *special* if it belongs to $\xi(T)$. The motivating example is the following: let Σ be an open Seifert fibered manifold. Then every incompressible torus is vertical (i.e., isotopic to a union of fibers); when two such tori are homotoped so as to intersect minimally, the intersection curves are isotopic to regular fibers of the Seifert fibration, which is unique except in a few special cases. Thus, in general, special curves should correspond to fibers of Seifert components of the JSJ-splitting we are looking for.

Let \mathcal{T} be a triangulation of M . We denote the k -skeleton of \mathcal{T} by $\mathcal{T}^{(k)}$. We say that \mathcal{T} has *bounded geometry* if there is a uniform upper bound on the number of simplices containing a given vertex. The *size* of a subset $A \subset M$ is the minimal number of 3-simplices of \mathcal{T} needed to cover A . This can be used to give a rough notion of distance (more precisely a *quasimetric* in the sense of [7]) $d_{\mathcal{T}}$ on M as follows: given two points $x, y \in M$, we let $d_{\mathcal{T}}(x, y)$ be the minimal size of a path connecting x to y , minus one. (Quasi)metric balls, neighborhoods, diameter etc. can be defined in the usual way.

If $\gamma \subset M$ is a curve which is in general position with respect to \mathcal{T} , then we define the *length* of γ as the cardinal of $\gamma \cap \mathcal{T}^{(2)}$. If F is a general position compact surface in M , then the *weight* of F is the cardinal of $F \cap \mathcal{T}^{(1)}$. We say that F is *normal* if it misses $\mathcal{T}^{(0)}$ and meets transversely each 3-simplex σ of \mathcal{T} in a finite collection of disks that intersect each edge of σ in at most one point. We say that F is a *least weight* surface if it has minimal weight among all surfaces isotopic to F .

Theorem 1.1. *Let M be an open, orientable, irreducible 3-manifold. Let \mathcal{T} be a triangulation of bounded geometry on M . Assume that the following hypotheses are fulfilled:*

- A) *There is a constant C_1 such that each isotopy class of canonical tori has a representative of weight at most C_1 .*
- B) *There is a constant C_2 such that for each least weight normal non-canonical incompressible torus T and each point $x \in T$, there is a special curve γ passing through x and of length at most C_2 .*
- C) *There are constants C_3, C_4 such that if T is a least weight normal non-canonical incompressible torus and σ a 3-simplex of \mathcal{T} such that $T \cap \sigma$ is disconnected, then there exists a normal torus T' of weight at most C_3 , which is not homotopically disjoint from T , and such that $T \cap T'$ contains a point whose distance from σ is at most C_4 .*
- D) *If T and T' are two disjoint least weight normal noncanonical tori, σ is a 3-simplex such that $T \cap \sigma$ (resp. $T' \cap \sigma$) consists of a single disk D (resp. D'), then one can find an annulus A connecting a special curve on T to a special curve on T' , and such that $\text{Int } A$ does not intersect T, T' , and that there is an essential arc $\alpha \subset A$ such that $\alpha \subset \sigma$ and α connects D to D' .*

Then M admits a JSJ-splitting.

Remarks.

- Condition A is an obvious way to rule out the phenomenon of Example M_3 in [6] where the pathology comes from canonical tori. One might wonder whether it is sufficient. The answer is probably no. Example \mathcal{O}_5 of the same paper is an *orbifold* with no canonical toric 2-suborbifolds and yet no JSJ-splitting (the definitions being extended to orbifolds in the obvious way.) It seems highly likely that manifolds with the same property exist.
- It is tempting to replace conditions B–D with simpler conditions, e.g. requiring that A be true even for noncanonical tori. However, such a hypothesis would be unreasonably strong in the sense that it would rule out examples as well as counterexamples. This is best explained by analogy with surfaces. Imagine that M is a 3-manifold with a JSJ-splitting, Σ a Seifert piece, and g is a complete Riemannian metric

whose restriction to Σ is obtained by lifting a metric on the base orbifold F . Then incompressible tori in Σ correspond to curves in F , with area corresponding to length. If one thinks that M is triangulated in such a way that the simplices are very small and of roughly uniform size, then the weight of the tori in Σ corresponds to the length of the curves in F , and two tori that intersect a common 3-simplex correspond to two curves which are very close.

Now to assume that every simple closed curve in F is homotopic to a curve of uniformly bounded length would be too restrictive: it would force F to be topologically finite. By contrast, it is reasonable to assume that for each long geodesic γ that comes very close to itself, there is a short geodesic intersecting essentially γ near the region where this happens. Condition C is analogous to this, and essentially says that there are ‘enough small tori’ to generate the Seifert pieces of the JSJ-splitting we are looking for. Likewise, conditions B and D respectively mean that the fibers of the Seifert pieces can be represented uniformly by small curves, and that different Seifert pieces are sufficiently far apart from one another.

We now give an informal outline of the proof of Theorem 1.1. It is somewhat oversimplified since we ignore such technical complications as Klein bottles and nonseparating tori, but should help the reader understand the main ideas. We use the Jaco-Rubinstein theory of PL minimal surfaces, which is reviewed in section 2.

If all incompressible tori in M are canonical, one takes a least PL area representative of each homotopy class. This gives a possibly infinite collection \mathcal{C} of canonical tori. Since members of \mathcal{C} are canonical, they are homotopically disjoint. Since they have least PL area in their respective homotopy classes, they are in fact disjoint. By hypothesis A, they have uniformly bounded weight. From this, it is not difficult to show that they have uniformly bounded diameter. Moreover, the collection \mathcal{C} is locally finite by Haken’s finiteness theorem. Hence the difficult case is when there are infinitely many noncanonical tori. In this case, we certainly do not expect PL least area tori to have uniformly bounded diameter. Hence they can accumulate, and the point is to understand how they accumulate.

More precisely, we need to use those tori to build Seifert submanifolds. For this, there is a well-known construction (cf., e.g., [10]): letting T, T' be two noncanonical tori intersecting minimally, one takes a regular neighborhood Σ of $T \cup T'$. This neighborhood has a fibration by circles such that $T \cap T'$ is a union of fibers. Its boundary consists of tori, which may or may not be incompressible. The compressible ones bound solid tori, to which the

fibration on Σ can be extended, possibly with exceptional fibers.

By iterating this construction, one builds a increasing sequence of Seifert submanifolds of M engulfing more and more noncanonical tori. The main difficulty is that while the union of those submanifolds is still Seifert fibered, the closure of this union need not be. In fact, it need even not be a manifold. This is where we use hypotheses B, C, and D.

The paper is organized as follows. In Section 2, we review normal surfaces and PL minimal surfaces. In Section 3 we review Seifert manifolds and prove Proposition 3.1, which justifies the definition of special curves. In Section 4 we deal with graph submanifolds and state a technical result, Theorem 4.1, from which Theorem 1.1 is easily deduced. Section 5 contains the proof of Theorem 4.1.

2 Normal surfaces and PL area

Recall from [7] the definition of a *regular Jaco-Rubinstein metric* on (M, \mathcal{T}) : it is a Riemannian metric on $\mathcal{T}^{(2)} - \mathcal{T}^{(0)}$ such that each 2-simplex is sent isometrically by barycentric coordinates to a fixed ideal triangle in the hyperbolic plane. The crucial property for applications to noncompact manifolds is that for every number n , there are finitely many subcomplexes of size n up to isometry. Let $F \subset M$ be a compact, orientable, embedded surface in general position with respect to \mathcal{T} . Recall from the introduction that the *weight* $\text{wt}(f)$ of f is the cardinal of $F \cap \mathcal{T}^{(1)}$. Its *length* $\text{lg}(F)$ is the total length of all the arcs in the boundaries of the disks in which F intersects the 3-simplices of \mathcal{T} . The *PL area* of F is the pair $|F| = (\text{wt}(F), \text{lg}(F)) \in \mathbf{N} \times \mathbf{R}_+$. We are interested in surfaces having least PL area among surfaces in a particular class with respect to the lexicographic order.

We collect in the next proposition some existence results and properties of PL least area surfaces. We are mostly interested in the case of tori and Klein bottles.

Proposition 2.1. *i. Let $F \subset M$ be an incompressible normal surface that is not homotopic to a double cover of an embedded nonorientable surface. Then there is a unique normal surface F_0 which is normally homotopic to F and has least PL area among such surfaces.*

ii. Let $F \subset M$ be an incompressible surface. Then there is a normal surface F_0 which has least weight among all surfaces homotopic to F . Furthermore, if F is not homotopic to a double cover of an embedded nonorientable surface, then there is a normal surface F_0 which has least PL area among all surfaces homotopic to F .

iii. Let $F, F' \subset M$ be incompressible surfaces that have minimal PL area in their respective homotopy classes. If F and F' are homotopically disjoint, then they are disjoint or equal. Otherwise, after a small perturbation they intersect minimally.

Proof. When M is compact, (i) follows from [3, Theorem 2], (ii) from [3, Theorems 3, 4, 6]; the first part of (iii) follows from [3, Theorem 7], and the second part, though not explicitly recorded in [3], can be proved by copying the proof of the corresponding result of [1] for Riemannian least area surfaces.

The proofs are easily extended to the noncompact case since the Jacobi-Rubinstein metric is regular, as noted in [7, Appendix A]. \square

In the sequel, we shall call “least PL area surface” an embedded normal surface that has least PL area in its homotopy class.

Remark. If an incompressible torus T is homotopic to a double cover of a Klein bottle embedded in M , then it bounds a submanifold of M homeomorphic to $K^2 \tilde{\times} I$. This implies that T is canonical. Hence the second sentence of Proposition 2.1(ii) always applies to noncanonical tori. If T is a canonical torus, however, there may be a least PL area torus homotopic to T , or a least PL area Klein bottle K such that T is homotopic to a double cover of K , or both.

In Section 5 we shall need a notion of normality for surfaces that are not properly embedded in M (arising as compact subsurfaces of noncompact normal surfaces.) We adopt the following definitions: let γ be a 1-submanifold of M . We say that it is *normal* if it is in general position with respect to \mathcal{T} , and for every 3-simplex σ of \mathcal{T} , each component of $\gamma \cap \sigma$ is an arc connecting two distinct faces of σ . (This notion is akin to M. Dunwoody’s *tracks*.) If F is a normal surface in M , and F' is a subsurface (with boundary) of F , then F' is *normal* if its boundary is normal. The notion of *normal homotopy* extends to normal curves in the obvious way.

The following lemma from [7] provides a useful inequality between the weight of a normal surface and the diameter of its image with respect to the quasimetric $d_{\mathcal{T}}$. The proof can be extracted from Lemmas 2.2 and A.1 of [7]. (There the surface F is supposed to be properly embedded, but the extension is immediate.)

Lemma 2.2. *Let F be a compact, not necessarily properly embedded, normal surface. Then $\text{diam}(F) \leq \text{wt}(F)^2$.*

It has the following important consequence:

Proposition 2.3. *Let \mathcal{C} be a collection of pairwise nonisotopic closed normal surfaces. If members of \mathcal{C} have uniformly bounded weight, then \mathcal{C} is locally finite.*

Proof. Let K be a compact subset of M . Let \mathcal{C}_K be the collection of members of \mathcal{C} that meet K . By Lemma 2.2, there is a uniform bound on the diameter of members of \mathcal{C}_K , which implies that they are all contained in some finite complex K' . Since there are only finitely many normal surfaces of given weight in K' up to isotopy, Proposition 2.3 follows. \square

3 Seifert submanifolds

We will use classical results on Seifert fiber spaces. For proofs we refer to [2] and [4].

We call a Seifert manifold *large* if its Seifert fibration is unique up to isotopy. Recall that if Σ is compact with nonempty incompressible boundary, then Σ is large unless Σ is homeomorphic to the thickened torus $T^2 \times I$ or the twisted I -bundle over the Klein bottle $K^2 \tilde{\times} I$.

Below is the proposition that justifies the definition of special curves on noncanonical tori.

Proposition 3.1. *Let T be an incompressible torus in M . Assume that T is not canonical, and not Seifert fibered. Then there is a unique free isotopy class $\xi(T)$ of simple closed curves in T such that if T' is an incompressible torus which is not homotopically disjoint from T and intersects T' minimally, then $T \cap T'$ consists of curves belonging to $\xi(T)$.*

Proof. Let T' be an incompressible torus which intersects T essentially and minimally. Then all components of $T \cap T'$ are essential on T . Since T is a torus, they are all freely isotopic. Let $\xi(T)$ be their free isotopy class. All we have to do is check that $\xi(T)$ does not depend on the choice of T' .

Let Σ' be a regular neighborhood of $T \cup T'$. Then Σ carries an obvious fibration such that $T \cap T'$ is a union of fibers. Since T, T' are incompressible in M , the generic fiber of this Seifert fibration is not contractible in M . The components of Σ' are tori T_1, \dots, T_k . Let T_i be one of them. Since T_i contains a fiber of the Seifert fibration, it is not contained in a 3-ball. Hence either T_i is incompressible in M or T_i bounds a solid torus V_i . In the latter case, since Σ' contains incompressible tori, V_i cannot contain Σ' , which implies that its interior is disjoint from Σ' .

Let Σ be the manifold obtained from Σ' by capping off the solid torus V_i for each compressible T_i . Since the generic fiber of Σ' is noncontractible in M ,

the Seifert fibration of Σ' extends to Σ . Furthermore, $\partial\Sigma$ is incompressible in M . Since M is open, $\partial\Sigma$ is not empty. Since Σ contains two nonisotopic incompressible tori, Σ must be large.

Let T'' be another incompressible torus meeting T essentially and minimally. Let η be a component of $T \cap T''$. Our goal is to show that $\eta \in \xi(T)$. For this it is enough to prove that η is freely homotopic to the generic fiber of Σ .

Let Σ'' be the complement of a regular neighborhood of T in Σ . Then Σ'' inherits a Seifert fibration. Call T_1, T_2 the two components of $\partial\Sigma''$ parallel to T . Set $U := T'' \cap \partial\Sigma''$. By an isotopy of T'' fixing η , we may assume that U consists entirely of essential curves. Call η_1, η_2 the two components of U parallel to η (with $\eta_i \subset T_i$).

Observe that $T \cap \Sigma''$ consists of finitely many annuli connecting the various components of U . Let A_1, A_2 be the annuli containing η_1, η_2 respectively. If at least one of these annuli are vertical in Σ'' , then we are done. Assume they are horizontal. Then Σ'' cannot be connected, for if it were, then it would have at least three boundary components, but such Seifert manifolds cannot contain horizontal annuli. Hence Σ'' has two components. Call Σ_i the one that contains T_i , for $i = 1, 2$. At least one of them, say Σ_1 , has more than one boundary component. Hence the base orbifold of Σ_1 must be a nonsingular annulus, which shows that $\Sigma_1 \cong T^2 \times I$. Now each component of $\Sigma_1 \cap T''$ is an incompressible annulus connecting T_1 to itself, so these annuli must be boundary-parallel. This contradicts the minimality of $T'' \cap T$. \square

Note that in the course of the proof we have shown:

Lemma 3.2. *[cf. [10, Lemma 3.2]] Let T, T' be two incompressible tori which intersect essentially and minimally. Let U be a regular neighborhood of $T \cup T'$. Then U is contained in some large Seifert submanifold Σ whose boundary is incompressible in M , and such that any special curve on T is isotopic to a fiber of Σ .*

A similar argument shows:

Lemma 3.3. *Let T be a noncanonical incompressible torus in M contained in some large Seifert submanifold Σ . Then any special curve on T is isotopic to a regular fiber of Σ .*

Next is a lemma that will enable us to enlarge a submanifold Σ of M so as to engulf all incompressible tori that are homotopic into Σ .

Lemma 3.4. *Let $\Sigma \subset M$ be a submanifold bounded by PL minimal incompressible tori. Let T be a component of $\partial\Sigma$. Assume that Σ is not a thickened torus. Then one of the following holds:*

- i. There is a submanifold X homeomorphic to $T^2 \times [0, \infty)$ such that $X \cap \Sigma = T$, or
- ii. All least area tori isotopic to T lie in Σ , or
- iii. There is an incompressible least PL area torus T' isotopic to T such that T and T' cobound a thickened torus X with $X \cap \Sigma = T$, and all least area tori isotopic to T lie in $\Sigma \cup X$, or
- iv. There is a submanifold X homeomorphic to $K^2 \tilde{\times} I$ such that $X \cap \Sigma = T$.

Proof. By Proposition 2.1(iii), least area incompressible tori are equal or disjoint. Thus if neither of cases (ii), (iii) and (iv) holds, then one can inductively construct an infinite sequence of least area tori T_n such that each T_n cobounds with T a thickened torus X_n , with the property that $X_n \cap \Sigma = T$, and $\Sigma \cup X$ does not contain all PL least area tori isotopic to T , and $X_n \subset X_{n+1}$. Since all T_n 's have the same weight, a compact subset of M can contain at most finitely of them, by Proposition 2.3. Hence the X_n exhaust a tame end of M and conclusion (i) holds. \square

4 Graph submanifolds

For the purposes of this paper, a *graph submanifold* of M is a 3-submanifold $\Sigma \subset M$ that contains a locally finite collection of pairwise disjoint, least weight normal canonical tori \mathcal{G} such that each component of Σ split along \mathcal{G} is Seifert fibered. This notion is needed in order to address a technical difficulty caused by nonseparating tori.

We now formulate the main technical result of this article, from which Theorem 1.1 will follow.

Theorem 4.1. *Under the hypotheses A, B, C and D, there exists a submanifold $\Sigma \subset M$ (possibly empty or equal to M) with the following properties :*

- i. Σ is a graph submanifold of M .
- ii. If T is a noncanonical least PL area incompressible torus in M , then $T \subset \Sigma$.
- iii. If F is a least PL area canonical torus or a least PL area Klein bottle, then either $F \subset \Sigma$ or $F \cap \Sigma = \emptyset$.

In Subsection 4.1, we show how to deduce Theorem 1.1 from Theorem 4.1. In Subsection 4.2 we introduce a notion of *taut* graph submanifold and prove an important technical result on the existence of such submanifolds.

4.1 Deduction of Theorem 1.1 from Theorem 4.1

Let Σ be a graph submanifold satisfying the conclusion of Theorem 4.1. Let \mathcal{G} be a locally finite collection of pairwise disjoint, least weight normal canonical tori \mathcal{G} such that each component of Σ split along \mathcal{G} is Seifert fibered.

For each isotopy class of canonical tori ξ , pick either a PL least area representative or a PL least area Klein bottle that is double covered by a torus of ξ ; this yields a collection \mathcal{C} of PL least area surfaces embedded in M . By Proposition 2.1 those surfaces are disjoint. Moreover, we may assume that for every $T \in \mathcal{G}$, if T is not homotopic to a double cover of a Klein bottle, then T is least PL area and $T \in \mathcal{C}$.

By hypothesis A and Proposition 2.3, \mathcal{C} is locally finite. By part (iii) of the conclusion of Theorem 4.1, \mathcal{C} is the disjoint union of four collections $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$, where \mathcal{C}_1 consists of the collection of tori in \mathcal{C}_1 that are contained in Σ , \mathcal{C}_2 consists of the collection of those that are disjoint from Σ , and \mathcal{C}_3 (resp. \mathcal{C}_4) consists of the Klein bottles that are contained in Σ (resp. are disjoint from Σ).

For each Klein bottle K in \mathcal{C}_3 , perturb K to a normal torus T that bounds a submanifold X homeomorphic to $K^2 \tilde{\times} I$, such that $X \subset \Sigma$. Since \mathcal{C} is locally finite, this can be done in such a way that all tori in the resulting collection \mathcal{C}'_3 are disjoint from one another and from members of \mathcal{C}_1 . As a consequence, each component of Σ split along $\mathcal{C}_1 \cup \mathcal{C}'_3$ is Seifert fibered.

Let Σ' be the Seifert submanifold of M obtained by taking all components of Σ split along $\mathcal{C}_1 \cup \mathcal{C}'_3$, and adding a “small” regular neighborhood of each member of $\mathcal{C}_2 \cup \mathcal{C}_4$. (Here “small” means that these neighborhoods should be disjoint from one another and from Σ . Again, this is possible because \mathcal{C} is locally finite.) Each component of Σ' is either a component of Σ split along $\mathcal{C}_1 \cup \mathcal{C}'_3$ or homeomorphic to $T^2 \times I$ or $K^2 \tilde{\times} I$, hence Σ' is a Seifert submanifold.

Claim. Every incompressible torus is homotopic into Σ' .

This is clear for canonical tori. Let T be a noncanonical torus. Then T can be isotoped into Σ by conclusion (ii) of Theorem 4.1. Since members of $\mathcal{C}_1 \cup \mathcal{C}'_3$ are canonical, T can in fact be isotoped into Σ' . This proves the claim.

In particular, every component of $M \setminus \Sigma'$ is atoroidal. This completes the proof of Theorem 1.1 assuming Theorem 4.1.

4.2 Taut graph submanifolds

Definition. Let $\Sigma \subset M$ be a graph submanifold. It is *taut* if it has incompressible boundary, no two components of $\partial\Sigma$ are parallel, every boundary

torus is PL least area, and any PL least area incompressible torus which can be homotoped into Σ is already contained in Σ .

Proposition 4.2. *i. If T, T' are noncanonical, PL least area incompressible tori which intersect essentially, then there exists a taut graph submanifold Σ containing both T and T' .*

ii. If Σ is a taut graph submanifold and T is a noncanonical PL least area incompressible torus, then there exists a taut graph submanifold Σ' containing both Σ and T .

Proof. (i) By Proposition 2.1(iii), one can perturb T and T' to intersect minimally. Then one takes a regular neighborhood Σ_1 of $T \cup T'$ and applies Lemma 3.2 to find a large, incompressible Seifert submanifold $\Sigma_2 \subset M$ containing Σ_1 . Then we can enlarge Σ_2 as follows:

For every pair of tori T_1, T_2 in $\partial\Sigma_2$ that are parallel outside Σ_2 , add a product region, obtaining a graph submanifold Σ_3 . Then for every boundary torus $T \subset \partial\Sigma_3$, add a region X as in Lemma 3.4 (in Case (ii) take $X = \emptyset$). We claim that the resulting submanifold Σ is a taut graph manifold. The only part of the definition that is not obvious is that any PL least area incompressible torus which can be homotoped into Σ lies in Σ . Let T_1 be such a torus, and let $T'_1 \subset \Sigma$ be a torus homotopic to T_1 . Since every component of $\partial\Sigma$ is PL least area, T_1 is disjoint from them by 2.1(iii). Hence if $T_1 \not\subset \Sigma$, then $T_1 \cap \Sigma = \emptyset$. This implies that T_1, T'_1 are disjoint, hence parallel. A product region between them must contain some component T_2 of $\partial\Sigma$. Thus T_1 is homotopic to T_2 , and must be contained in Σ .

(ii) We distinguish three cases.

If $T \subset \Sigma$ we can simply put $\Sigma' := \Sigma$.

If T intersects essentially some component T' of $\partial\Sigma$, then after perturbation of T , $T \cap T'$ is a union of special curves. Let Σ_1 be a regular neighborhood of $\Sigma \cup T$. Then Σ_1 is Seifert fibered. The rest of the proof is as in (i).

If $T \cap \Sigma = \emptyset$, let T' be a least PL area incompressible torus intersecting T essentially. If $T' \cap \Sigma \neq \emptyset$, then using the previous case we can find a taut graph submanifold Σ'' containing $\Sigma \cup T'$, and then we are reduced to the first two cases. Otherwise we let Σ_1 be the union of Σ and a regular neighborhood of $T \cup T'$, and argue as in (i). \square

5 Proof of Theorem 4.1

5.1 An increasing sequence of taut graph submanifolds

Proposition 5.1. *There exists a (finite or infinite) sequence $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \dots \subset \Sigma_n \subset \dots$ of taut graph submanifolds of M such that any noncanonical, least PL area incompressible torus is contained in some Σ_n . Furthermore, each Σ_n is the union of a compact submanifold with a finite (possibly empty) collection of submanifolds homeomorphic to $T^2 \times [0, \infty)$; in particular, $\partial\Sigma_n$ has finitely many connected components.*

Proof. Let $\mathcal{L} = [T_1], [T_2], \dots$ be a list of all homotopy classes of noncanonical incompressible tori. The construction is by induction using Proposition 4.2.

Set $\Sigma_0 := \emptyset$. If the list \mathcal{L} is nonempty, take a least PL area representative of the first class $[T_1]$. Since it is noncanonical, another incompressible torus intersects T_1 essentially. Proposition 4.2(i) gives a submanifold Σ_1 . Then, assuming we have constructed $\Sigma_1, \dots, \Sigma_n$, we take a least PL area representative of $[T_{n+1}]$ and apply Proposition 4.2(ii). \square

From now on we fix a sequence Σ_n satisfying the conclusion of Proposition 5.1. If Σ_n is finite or eventually constant, then Theorem 4.1 follows immediately by taking $\Sigma := \bigcup \Sigma_n$. Henceforth we assume (by going to a subsequence) that Σ_n is strictly increasing.

Let σ be a 3-simplex of \mathcal{T} . Then $\sigma \cap \bigcup_n \partial\Sigma_n$ is a disjoint union of normal disks. We say that two such disks D, D' are *equivalent* if they are normally homotopic and there exists n and a component X of $\Sigma_n \cap \sigma$ that contains both D and D' . This relation is obviously reflexive and symmetric. Since the sequence Σ_n is increasing, it is also transitive. Hence it is an equivalence relation.

Lemma 5.2. *For each 3-simplex σ , only finitely many components of $\bigcup_n \partial\Sigma_n$ that intersect σ can belong to canonical tori.*

Proof. This follows immediately from Hypothesis A and Proposition 2.3. \square

Lemma 5.3. *For each 3-simplex σ , only finitely many components of $\bigcup_n \partial\Sigma_n$ have disconnected intersection with σ .*

Proof. Assume the contrary. Let T_1, \dots, T_k, \dots be an infinite sequence of tori of $\bigcup_n \partial\Sigma_n$ having disconnected intersection with σ . Since each Σ_n has only finitely many boundary components, we may assume by taking a subsequence that there is a sequence $n(k)$ going to infinity with k , such that for each k , $T_k \subset \partial\Sigma_{n(k)}$ but $T_k \not\subset \partial\Sigma_{n(k)-1}$.

Apply hypothesis C to each T_k , getting a torus T'_k . Then the T'_k 's have uniformly bounded diameter, and their distances from σ are uniformly bounded. Hence there are only finitely many of them up to normal homotopy.

It follows that some T'_{k_0} intersects essentially infinitely many of the T_k 's. Then T'_{k_0} is noncanonical, hence contained in some Σ_{n_0} . This is a contradiction. \square

Proposition 5.4. *For each 3-simplex σ , there are only finitely many equivalence classes of disks in $\sigma \cap \bigcup_n \partial\Sigma_n$.*

Proof. The proof is by contradiction. Suppose that there is an infinite collection D_k of normally homotopic disks in $\sigma \cap \bigcup_n \partial\Sigma_n$ that are not pairwise equivalent. By Lemma 5.3, one can assume without loss of generality that all those disks, as well as all disks that are equivalent to them, belong to different tori of $\bigcup_n \partial\Sigma_n$. Moreover, we may assume that all those tori have connected intersection with σ , so that hypothesis D can be applied to them. We also assume using Lemma 5.2 that all those tori are noncanonical.

Let T_1, T_2 be such that $D_1 = T_1 \cap \bigcup_n \partial\Sigma_n$ and $D_2 = T_2 \cap \bigcup_n \partial\Sigma_n$.

Lemma 5.5. *There exists n and a connected Seifert submanifold Σ'_n of Σ_n such that $T_1 \subset \Sigma'_n$ and $T_2 \subset \Sigma'_n$.*

Proof. By hypothesis, T_1 and T_2 are PL least area and noncanonical, so there exists n_1, n_2 such that $T_i \subset \Sigma_{n_i}$ for $i = 1, 2$. Setting $n := \max(n_1, n_2)$ and using the fact that the sequence Σ_n is increasing, one has $T_i \subset \Sigma_n$. Let Σ'_1, Σ'_2 be the maximal Seifert components of the graph manifold Σ_n containing T_1, T_2 respectively. If $\Sigma'_1 = \Sigma'_2$, then the lemma is proved. Otherwise $\Sigma'_1 \cap \Sigma'_2$ is the empty set.

By hypothesis D, there is an annulus A connecting a special curve on T_1 to a special curve on T_2 . By a homotopy, we can ensure that A is a union $A_1 \cup_{c_1} A_2 \cup_{c_2} \cdots \cup_{c_{k-1}} A_k$, where each c_j is an essential simple closed curve on A , and the A_j 's are either contained in $\Sigma'_1 \cup \Sigma'_2$ or have interior disjoint from that set. Furthermore, we may assume that each A_j that is contained in $\Sigma'_1 \cup \Sigma'_2$ is vertical.

Choose j such that A_j has interior disjoint from $\Sigma'_1 \cup \Sigma'_2$ and connects some component T'_1 of $\partial\Sigma'_1$ to some component T'_2 of $\partial\Sigma'_2$. Since the Seifert manifold Σ'_1 is large, its base orbifold contains an essential properly embedded arc α_1 connecting the projection of T'_1 to itself. Hence Σ'_1 contains an essential annulus A'_1 connecting T'_1 to itself. Likewise, Σ'_2 contains an essential annulus A'_2 connecting T'_2 to itself. By patching together A'_1, A'_2 , and two parallel copies of A_j , we get a torus T which is vertical in some Seifert manifold Σ containing both Σ'_1 and Σ'_2 . By construction, T is incompressible and is not

homotopic into either Σ'_1 or Σ'_2 . Hence it is noncanonical. Still denote by T a least PL area representative of the homotopy class of T . Then there exists $n' > n$ such that $T \subset \Sigma_{n'}$.

It follows that some connected Seifert submanifold of $\Sigma_{n'}$ contains $\Sigma'_1 \cup \Sigma'_2$, and therefore the original tori T_1 and T_2 as well. \square

Having proved Lemma 5.5, we continue the proof of Proposition 5.4. Let $X \subset \sigma$ be the product region between D_1 and D_2 . If $X \subset \Sigma'$, then D_1 and D_2 are equivalent, contradicting our hypothesis. Thus $\partial\Sigma'_n$ intersects X in at least two nonequivalent normal disks D'_1, D'_2 . Furthermore, the components T'_1, T'_2 of $\partial\Sigma'_n$ containing D'_1, D'_2 respectively are different. Let X' denote the product region between D'_1 and D'_2 . Without loss of generality we assume that $X' \cap \Sigma'_n = \emptyset$.

By hypothesis D, there is an annulus A connecting a special curve of T'_1 to a special curve of T'_2 and containing an arc $\alpha \subset X'$, essential on A , and connecting D'_1 to D'_2 . Since Σ' is connected, there is an arc $\alpha' \subset \Sigma'$ connecting the endpoints of α . Thus $\alpha \cup \alpha'$ is a simple closed curve intersecting each of T'_1 and T'_2 in a single point. Since Σ'_n is Seifert fibered and ∂A consists of special curves, there is a properly embedded annulus $A' \subset \Sigma'$ that connects both components of ∂A , and we may assume that $\alpha' \subset \Sigma'$.

Let T be the union of A and A' . Then T is an embedded, incompressible torus, that intersects T'_1 and T'_2 essentially. In particular, it is noncanonical. Hence there exists $n' > n$, a Seifert submanifold $\Sigma'_{n'} \subset \Sigma_{n'}$ and a torus T' homotopic to T such that $\Sigma'_n \subset \Sigma'_{n'}$ and $T' \subset \Sigma'_{n'}$.

If $\Sigma'_{n'}$ contains X' , then D'_1 and D'_2 are equivalent, contradicting an assumption made earlier. Hence there is a torus $T'' \subset \partial\Sigma'_{n'}$ such that $T'' \cap X'$ is a disk D'' . Now α intersects D'' in an odd number of points, and $\alpha' \cap T'' = \emptyset$ because $\alpha' \subset \Sigma'_n \subset \Sigma'_{n'}$. Hence $\alpha \cup \alpha'$ is not homotopically disjoint from T'' . Since T' is homotopic to T , it contains a curve homotopic to $\alpha \cup \alpha'$. This implies that T' and T'' intersect essentially, which contradicts the fact that T'' is a boundary component of some submanifold containing T' . \square

Modify Σ_n in the following way: for each edge e of \mathcal{T} we look at the set $S_e := e \cap \bigcup_n \partial\Sigma$. We can define an equivalence relation on S_e by saying that two points x, y are equivalent if for some 3-simplex σ containing e there are equivalent disks D, D' in $\sigma \cap \bigcup_n \partial\Sigma_n$ such that $D \cap e = x$ and $D' \cap e = x'$. By Proposition 5.4, this relation also has only finitely many classes.

For every equivalence class c , let I_c denote the closure of the convex hull of c . Here we encounter a technical difficulty: perhaps the I_c 's are not disjoint, so the closure of the union of the Σ_n is not a submanifold, because some points are approached from both sides.

To deal with this, choose arbitrarily a segment I'_c contained in the interior of I_c . Then map linearly I_c onto I'_c . Let c' be the image of c under this mapping. From the union of all those c 's, we can, by taking convex hulls in each 3-simplex, reconstruct a collection of normal tori bounding graph submanifolds. Each of these new tori is normally homotopic to an old one. We keep the same notation.

Let Σ be the closure of the union of the Σ_n . Then Σ is a submanifold of M , whose interior is the union of the Σ_n . By construction, it satisfies properties (ii) and (iii) of the conclusion of Theorem 4.1. The remaining task is to show that Σ is a graph submanifold of M .

5.2 End of the proof

In this subsection, we are interested in the boundary components of Σ . We shall prove that they are incompressible annuli or canonical tori. In order to study them, we need to make sense of the notion that they are approximated by boundary components of the Σ_n 's, which are normal tori. To this effect we give the following definition:

Definition. We say that a sequence F_n of normal surfaces *converges to* a normal surface F if the following requirements are fulfilled:

- i. For any compact normal subsurface $K \subset F$ (in the sense of Section 2), there exists $n_0(K)$ such that for all $n \geq n_0(K)$, there is a normal subsurface $K_n \subset F_n$ normally homotopic to K ;
- ii. If $K_n \subset F_n$ is any sequence of compact normal subsurfaces having the property that for all sufficiently large n, n' , K_n is normally homotopic to $K_{n'}$, then there exists a subsurface $K \subset F$ which is normally homotopic to K_n for all sufficiently large n .

The following lemma is immediate from the construction in the previous subsection:

Lemma 5.6. *Let F be a component of $\partial\Sigma$. There is a sequence T_n of tori such that for every n , T_n is a component of $\partial\Sigma_n$, and which converges to F .*

We next record an important consequence of hypothesis B:

Lemma 5.7. *There is a constant C'_2 such that if T is a noncanonical normal least weight torus in M , and D is a normal subdisk of T , then $\text{diam}(D) \leq \text{diam}(\partial D) + C'_2$.*

Proof. Apply hypothesis B to a point $x \in D$ whose distance to ∂D is maximal, noting that a special curve is noncontractible in M , hence cannot lie entirely in D . \square

Lemma 5.8. *The following assertions hold:*

- i. F is incompressible in M ;
- ii. F is a torus or an annulus;
- iii. If F is a torus, then it is canonical.

Proof. (i) We argue by contradiction. Let D be a compressing disk for F whose boundary is in general position. By a sequence of isotopies that reduce the length, we may assume that ∂D is normal in the sense of section 2. Now ∂D is contained in some compact normal subsurface K of F . Applying Lemma 5.6 and using part (i) of the definition of convergence of sequences of normal surfaces, we see that there is for n large enough a normal subsurface $K_n \subset T_n$ normally homotopic to K . Hence we can find a normal curve $\gamma_n \subset K_n$ which is connected to ∂D by a small annulus A_n , and we find compression disks $D_n := A_n \cup D$ for T_n . Since each T_n is incompressible, there is for each large n a disk $D'_n \subset T_n$ with $\partial D_n = \partial D'_n$.

By construction the diameter of $\partial D'_n$ is independent of n . Hence Lemma 5.7 gives a uniform upper bound for the diameters of the D'_n 's. By Lemma 2.2, the weights of the D'_n are also uniformly bounded above. Hence there are only finitely many of them up to normal homotopy. Using part (ii) of the definition of convergence of sequences of normal surfaces, we get a disk $D' \subset F$ such that $\partial D = \partial D'$.

(ii) If F is compact, then Lemma 5.6 implies that F is a torus. Hence we suppose that F is noncompact.

Let $\gamma \subset F$ be a simple closed curve. We say that it is *special* if it is normally homotopic to a special curve $\gamma_n \subset T_n$ for large n . In particular, any special curve in F is essential in M , hence essential in F .

Sublemma 5.9. *There is a constant C_2'' such that for every $x \in F$, there is a special curve $\gamma \subset F$ of length at most C_2'' whose distance to x is at most C_2'' .*

Proof. Take $x \in F$. Then x is a limit of a sequence $x_n \in T_n$. By hypothesis B of the main theorem, through each x_n there is a special curve $\gamma_n \subset T_n$ of length at most C_2 . For each n we perform a normalizing sequence of homotopies, getting a normal special curve $\gamma'_n \subset T_n$. The bound on the length of γ_n gives bounds on both the length of γ'_n and the distance between

γ'_n and x_n . Hence there are only finitely many γ'_n 's up to normal isotopy. This shows that a subsequence of γ'_n converges to some normal special curve $\gamma \subset F$ whose length and distance from x can be bounded above by a constant C''_2 depending only on C_2 . This proves Sublemma 5.9. \square

Next we show that F is planar. By way of contradiction, suppose that F is nonplanar. Then it contains a nonplanar compact normal subsurface K . Let $K_n \subset T_n$ be a sequence of approximating subsurfaces. Then for each n , K_n has genus 1. Hence its boundary consists of curves of uniformly bounded length that bound disks on T_n . By Lemma 5.7, those disks have uniformly bounded diameter. This implies that the T_n 's have uniformly bounded diameter, contradicting the noncompactness of the limit F . This contradiction proves that F is planar.

By Sublemma 5.9, F contains an essential curve, so it cannot be homeomorphic to \mathbf{R}^2 . If it had more than two ends, then we could find a compact subsurface $K \subset F$ such that $F \setminus K$ has at least three noncompact components U_1, U_2, U_3 . For $i = 1, 2$ pick a point $x_i \in U_i$ such that $d(x_i, K) > C_2$. By Sublemma 5.9, there are special curves $\gamma_i \subset U_i$ for $i = 1, 2$. Now special curves are homotopic in F , as can be seen by approximation in some T_n . This contradiction proves that F has two ends. Hence it is an annulus and (ii) is proven.

(iii) If F is a torus, then for n sufficiently large, $\partial\Sigma$ contains a torus T_n normally homotopic to F . By uniqueness of the least PL area representative of a normal homotopy class, the T_n 's are equal. If they were not canonical, then there would exist a least PL area T' that is not homotopically disjoint from T_n . For large n , T' would have to be contained in Σ_n . This is a contradiction. \square

At last we show that Σ is a graph submanifold of M : let Z be a union of PL least area canonical tori and Klein bottles such that every canonical torus contained in Σ is homotopic to some component of Z , or a double cover of some component of Z . By hypothesis A and Proposition 2.3, Z is a submanifold. Let U be a regular neighborhood of Z in Σ . All we have to show is that every component of $\Sigma \setminus U$ is Seifert fibered.

Let X be such a component. Choose for each annular component A_i of ∂X an annulus $A'_i \subset X$ properly homotopic to A_i , in such a way that the A'_i 's do not intersect one another, and that for each n , $A'_i \cap \Sigma_n$ is empty or an annulus whose core is a special curve. Then the A'_i 's together with the toral components of ∂X bound a submanifold $X' \subset X$ such that X retraction deforms onto X' . In particular, X and X' are homeomorphic, so it is enough to prove that X' is Seifert fibered.

In order to do this, we notice that the submanifolds $\Sigma_n \cap X'$ can be given compatible Seifert fibrations such that for every annulus A'_i and every n , if $A'_i \cap \Sigma_n$ is nonempty, then it is a vertical annulus. This gives a Seifert fibration on $X \setminus \bigcup_i A_i$ such that every A'_i is vertical. This Seifert fibration restricts to a Seifert fibration on X' . Hence Σ is a graph submanifold of M , and the proof of Theorem 4.1 is complete.

References

- [1] M. Freedman, J. Hass, and P. Scott. Least area incompressible surfaces in 3-manifolds. *Invent. Math.*, 71:609–642, 1983.
- [2] W. Jaco. *Lectures on three-manifold topology*, volume 43 of *CBMS Lecture Notes*. American Mathematical Society, Providence, Rhode Island, USA, 1980.
- [3] W. Jaco and J. H. Rubinstein. PL minimal surfaces in 3-manifolds. *J. Differential Geom.*, 27(3):493–524, 1988.
- [4] W. H. Jaco and P. B. Shalen. *Seifert fibered spaces in 3-manifolds*. Number 220 in *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, Rhode Island, USA, Sept. 1979.
- [5] K. Johannson. *Homotopy equivalences of 3-manifolds with boundary*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.
- [6] S. Maillot. Some open 3-manifolds and 3-orbifolds without locally finite canonical decompositions. arXiv:0802.1438.
- [7] S. Maillot. Open 3-manifolds whose fundamental groups have infinite center, and a torus theorem for 3-orbifolds. *Trans. Amer. Math. Soc.*, 355(11):4595–4638 (electronic), 2003.
- [8] S. Maillot. A spherical decomposition for Riemannian open 3-manifolds. *Geom. Funct. Anal.*, 17(3):839–851, 2007.
- [9] W. D. Neumann and G. A. Swarup. Canonical decompositions of 3-manifolds. *Geom. Topol.*, 1:21–40 (electronic), 1997.
- [10] P. Scott. There are no fake Seifert fibered spaces with infinite π_1 . *Annals of Math.*, 117:35–70, 1983.

Institut de Recherche Mathématique Avancée, Université Louis Pasteur,
7 rue René Descartes, 67084 Strasbourg Cedex, France
`maillot@math.u-strasbg.fr`